# THE EFFECT OF PHASE IN THOMSON SCATTERING OF A HIGH INTENSITY BEAM

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**ABSTRACT.** In this paper the effect of phase of the incident high intensity radiation in Thomson scattering, is investigated. It is shown that, it produces broadening of the scattered line superimposed on the mean shift. The other important result is that the intensity dependent part of the cross-section cancels out, on averaging over the phase, so that the Thomson formula for the cross-section is still valid for the next higher order of intensity.

## INTRODUCTION

In a recent paper the author (Sen Gupta 1966, subsequently this will be refered as I) has studied the effect of intensity on Thomson scattering with radiation reaction. For simplicity the phase of the incident radiation was taken to be zero in I. Presently Prakash (1967) in a note has discussed the effect of phase on the scattered frequency and the cross-sections. Though most his remarks are irrelevant to the physical problem, because of the unphysical nature of the electro-magnetic field he has considered, yet it is worth-while to investigate the effect of the phase. The object of this paper is to study the same problem as in I, but with an arbitrary phase $\delta$ of the incident radiation. Let the directed monochromatic beam of (circular) frequency $\omega_0$ along the direction $\vec{n}$ be described by the vector potential,

$$\boldsymbol{a} = \boldsymbol{j}\, a \cos(\omega_0\theta+\delta);\ \theta = t - \frac{(\boldsymbol{n}\cdot\boldsymbol{r})}{c} \tag{1}$$

Since the only difference with I is that the phase $\delta$ which was taken as zero in I, there cannot be any qualitative difference from I. The procedure for solving the equation of motion and the determination of scattered frequencies and the cross-sections of scattering, are exactly the same. Hence, neither the procedure nor the usual observations are repeated. But only important steps and modified expressions are reported here.

In the next section the solution of the Lorentz equation of motion with radiation reaction, is indicated. As before $\Sigma$, the reference system, is the one in which the electron is initially at rest. Section III is devoted to the investigations of the effect of phase on the shift of the scattered frequencies which depends on intensity and the scattering cross-sections. The shift of the frequency is isotropic but depends on the phase $\delta$, hence the net result is to broaden the scattered line in addition to a mean shift of frequency due to intensity. The cross-section for scattering with the fundamental frequency also depends on the phase but it is very interesting to note that on averaging over the phase, the scattering cross-section remains the same upto terms first order in $\xi = e\,a/mc^2$. The initial phase has no effect on the cross-section for the first harmonic.

## THE SOLUTION OF THE EQUATION OF MOTION

The equation of motion* (Sen Gupta, 1967) is

$$\frac{d\boldsymbol{V}}{dt}-\frac{\mu}{\omega_0}\frac{d^2\boldsymbol{V}}{dt^2}=\omega_0 c\,\xi\left[\boldsymbol{j}+\frac{\boldsymbol{V}}{c}\times(\boldsymbol{n}\times\boldsymbol{j})\right]\sin(\omega_0\theta+\delta) \qquad .. \quad (2)$$

where $\mu=\dfrac{2}{3}\dfrac{e^2\omega_0}{mc^3}$. It has already been shown in I, that the relativistic corrections are not effective to the order of magnitude in which we are interested, namely upto quadratic in $\xi$ and linear in $\mu$. As before we take the solution in the form

$$\frac{\omega}{c}\boldsymbol{r}=\boldsymbol{\alpha}(\sin\tau-\tau)+\boldsymbol{\beta}(\sin 2\tau-2\tau)$$
$$+\boldsymbol{\alpha}'(1-\cos\tau)+\boldsymbol{\beta}'(1-\cos 2\tau)+\xi^2\boldsymbol{R}(\tau) \qquad (3)$$

where $\tau=\omega t$. The expressions for velocity $\boldsymbol{V}$ and $\theta\omega_0$ are,

$$\frac{\boldsymbol{V}}{c}=\boldsymbol{\alpha}(\cos\tau-1)+2\boldsymbol{\beta}(\cos 2\tau-1)$$
$$+\boldsymbol{\alpha}'\sin\tau+2\boldsymbol{\beta}'\sin 2\tau+\xi^2\frac{d\boldsymbol{R}}{d\tau} \qquad .. \quad (4)$$

$$\omega_0\theta=\omega_0\{1+(\boldsymbol{n}.\boldsymbol{\alpha})+2(\boldsymbol{n}.\boldsymbol{\beta})\}t-\frac{\omega_0}{\omega}\{(\boldsymbol{n}\,\boldsymbol{\alpha})\sin\tau+(\boldsymbol{n},\boldsymbol{\beta})\sin 2\tau+(\boldsymbol{n}.\boldsymbol{\alpha}')(1-\cos\tau)$$
$$+(\boldsymbol{n}.\boldsymbol{\beta}')(1-\cos 2\tau)+\xi^2(\boldsymbol{n}.\boldsymbol{R})\}. \qquad .. \quad (5)$$

Hence the expression for $\omega$ is

$$\omega=\omega_0\{1+(\vec{\boldsymbol{n}}.\vec{\boldsymbol{\alpha}})+2(\vec{\boldsymbol{n}}.\vec{\boldsymbol{\beta}})\}. \qquad .. \quad (6)$$

These expressions (3)—(6) are formally the same as I, but $\boldsymbol{\alpha}$'s and $\boldsymbol{\beta}$'s now depend on $\delta$, due to the presence of $\delta$ on the right hand side of eq. (2). From eq. (5)

$$\xi\sin(\omega_0\theta+\delta)=\xi[\sin(\tau+\delta)-\tfrac{1}{2}(\mathbf{n}.\boldsymbol{\alpha})\{\sin(2\tau+\delta)-\sin\delta\}$$
$$-\tfrac{1}{2}(\mathbf{n}.\boldsymbol{\alpha}')\{2\cos(\tau+\delta)-\cos(2\tau+\delta)-\cos\delta\}]$$

*It deserves to be mentioned that the solution of the Lorentz equation of motion, without damping, was reported much earlier by Frenkel. (1925).

The equations for determining $\vec{\boldsymbol{\alpha}}$'s and $\vec{\boldsymbol{\beta}}$'s are now given by

$$\vec{\boldsymbol{\alpha}}-\mu\vec{\boldsymbol{\alpha}} = \xi\,[\cos\delta\{-\boldsymbol{j}+\boldsymbol{\alpha}\times(\boldsymbol{n}\times\boldsymbol{j})\}-\sin\delta\,\boldsymbol{j}(\boldsymbol{n}.\boldsymbol{\alpha}')] \quad .. \quad (8)$$

$$\boldsymbol{\alpha}'+\mu\boldsymbol{\alpha} = \xi[\sin\delta\{\boldsymbol{j}-\vec{\boldsymbol{\alpha}}\times(\vec{\boldsymbol{n}}\times\vec{\boldsymbol{j}})\}-\cos\delta\,\boldsymbol{j}(\boldsymbol{n}.\boldsymbol{\alpha}')] \quad .. \quad (9)$$

$$\boldsymbol{\beta}-2\mu\boldsymbol{\beta}' = -\frac{\xi}{8}\,[\sin\delta\{-(\boldsymbol{n}.\boldsymbol{\alpha}')\boldsymbol{j}+\boldsymbol{\alpha}'\times(\boldsymbol{n}\times\boldsymbol{j})\}$$

$$+\cos\delta\{(\boldsymbol{n}.\boldsymbol{\alpha})\boldsymbol{j}+\boldsymbol{\alpha}\times(\boldsymbol{n}\times\boldsymbol{j})\}] \quad .. \quad (10)$$

$$\boldsymbol{\beta}'+2\mu\boldsymbol{\beta} = \frac{\xi}{8}\,[\sin\delta\{(\boldsymbol{n}.\boldsymbol{\alpha})\boldsymbol{j}+\boldsymbol{\alpha}\times(\boldsymbol{n}\times\boldsymbol{j})$$

$$+\cos\delta\{(\boldsymbol{n}.\boldsymbol{\alpha}')\boldsymbol{j}-\boldsymbol{\alpha}'\times(\boldsymbol{n}\times\boldsymbol{j})\}], \quad .. \quad (11)$$

and

$$\xi^2\left(\frac{d^2\vec{\boldsymbol{R}}}{d\tau^2}-\mu\frac{d^3\vec{\boldsymbol{R}}}{d\tau^3}\right)$$

$$= \xi[\boldsymbol{j}\{(\boldsymbol{n}.\boldsymbol{\alpha}')\cos\delta+(\boldsymbol{n}.\boldsymbol{\alpha})\sin\delta\}+(\boldsymbol{\alpha}\sin\delta+\vec{\boldsymbol{\alpha}}'\cos\delta)\times(\boldsymbol{n}\times\boldsymbol{j})]. \quad .. \quad (12)$$

The eqs. (8)-(11) may be solved easily. Thus

$$\boldsymbol{\alpha} = -\xi(\cos\delta-\mu\sin\delta)\{\boldsymbol{j}+\xi\,\boldsymbol{n}(\cos\delta-\mu\sin\delta\} \quad .. \quad (13)$$

$$\boldsymbol{\alpha}' = \xi(\sin\delta+\mu\cos\delta)\{\boldsymbol{j}+\boldsymbol{n}(\cos\delta-\mu\sin\delta)\} \quad .. \quad (14)$$

$$\boldsymbol{\beta} = \frac{\xi^2}{8}\,\boldsymbol{n}\,(\cos 2\delta-3\mu\sin 2\delta) \quad .. \quad (15)$$

$$\boldsymbol{\beta}' = -\frac{\xi^2}{8}\,\boldsymbol{n}\,(\sin 2\delta+3\mu\cos 2\delta), \quad .. \quad (16)$$

Substituting these values of $\boldsymbol{\alpha}$'s in eq. (12), it reduces to

$$\frac{d^2\boldsymbol{R}}{d\tau^2}-\mu\frac{d^3\vec{\boldsymbol{R}}}{d\tau^3} = \frac{\mu}{2}\,\boldsymbol{n}, \quad (17)$$

which is independent of the phase, It is the same as in I. Hence, to the order of approximation we are interested, the small acceleration due to radiation reaction is not influenced by the phase. With these expressions for $\boldsymbol{\alpha}$'s and $\boldsymbol{\beta}$'s the eqs. (6) and (7) determine the position and velocity. It is quite clear that there is no qualitative change in the nature of motion. The only effect of the phase $\delta$ is to introduce relative phase differences between the various harmonics in a complicated manner.

## FREQUENCY SHIFT AND SCATTERING CROSS-SECTIONS

(i) *The frequency of the scattered radiation* :

The frequency of the radiation emitted is the mechanical frequency of the electron, which are multiples of the fundamental $\boldsymbol{\omega}$. From eqs. (6), (13) and (15),

$$\omega = \omega_0 \left[ 1 - \frac{\xi^2}{2} \{1+\tfrac{1}{2}(\cos 2\delta - \mu \sin 2\delta)\} \right]. \tag{18}$$

Thus the scattered frequency, in $\Sigma$, is isotropic but it depends also on the phase $\delta$. The fundamental scattered frequencies lie between

$$\omega_{\max} = \omega_0 \left(1 - \frac{\xi^2}{4}\right) \text{ and } \omega_{\min} = \omega_0 \left(1 - \frac{3\xi^2}{4}\right). \tag{19}$$

Since $\delta$ may take arbitrary values and the observations are made on a system of electrons the scattered radiation is no longer a sharp monochromatic line but a broadened line of width $\xi^2/2$ and the mdan position is given by,

$$\omega_{\text{mean}} = \omega_0(1 - \xi^2/2). \tag{20}$$

It may be noted here that this is exactly the classical limit of frequency shift; (in the transverse direction) of the corresponding expression obtained from the quantum theory (Sen Gupta, 1952, 1967; Goldman, 1964),

(ii) *Scattering Cross-sections*

The expressions for the cross-section for scattering may be obtained in the usual manner.

(a) *Scattering with fundamental frequency*

$$\frac{d\sigma(\delta)}{d\Omega} - \left[ 1-(\boldsymbol{k}.\boldsymbol{j})^2 + 4\xi(\vec{\boldsymbol{k}}.\vec{\boldsymbol{j}}) \left\{ 1-(\boldsymbol{k}.\boldsymbol{j})^2 - \frac{(\boldsymbol{k}.\boldsymbol{n})}{2} \right\} (\cos\delta - \mu \sin\delta) \right].. \tag{21}$$

where $r_0$ is the classical radius of the electron and $\vec{\boldsymbol{k}}$ is the unit vector along the direction of observation. Since the phase of the initial radiation is arbitrary, we can take their distribution to be uniform, i.e. equal range of values are equally probable. The observed cross-section is the average over all probable values of $\delta$. Thus

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi} \int^{2\pi} \frac{d\sigma(\delta)}{d\Omega} d\delta$$

$$= r_0^2\{1-(\boldsymbol{k}.\boldsymbol{j})^2\}. \tag{22}$$

This is the well-known Thompson scattering cross-section. It is vey important to note that $\xi$ dependent term in the expression for cross-section cancel out on averaging over the phase of the incident radiation and the usual expression for the scattering cross-section, which is independent of the phase, is valid upto this order. This is because of the fact that in eq. (21) only ($\cos\delta-\mu\sin\delta$) appears. Thus second term of the right hand side of eq. (21) in I, is to be dropped out.

(b) *Scattering with first harmonics*

The cross-section for scattering with frequency $2\omega$ being proportional to $|\boldsymbol{k}\times\boldsymbol{\beta}^2|+|\boldsymbol{k}\times\boldsymbol{\beta}'|^2$ is independent of $\delta$ as can be seen easily from eqs. (15, and (16). Thus the cross-section for first harmonies remains the same as given by eqs. (2) and (23) in I.

Finally since the acceleration does not change due to the phase the nature of "white radiation" remains the same.

DISCUSSION

It is quite clear that the phase of the incident radiation cannot change the qualitative nature of the problem. This is because of the fact that $\delta$ in eqn (1) may be made zero by suitable translation of space and time. This translation will only change the initial values as the initial velocity in general is not zero in that case. We can summarize the two main contribution of the phase of the initial radiation :

(i) it produces broadening of the scattered line having the width $\omega_0\xi^2/2$ and mean at $\omega_0(1-\xi^2/2)$ for the fundamental and both these expressions are multiplied by 2 for the first harmonic, the frequency of the scattered radiation is also isotropic in $\Sigma$;

(ii) the contribution of the cross-section which depends on intensity cancel out on avraging over the phase leading to the validity of Thomson formula in this order.

(iii) lastly the cross-section for the first harmonics is not altered.

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